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Jubileuszowy Zjazd
Matematyków Polskich
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UNIQUENESS, REGULARITY
OF SOLUTIONS TO PDES
AND REAL WORLD PROB-
LEMS

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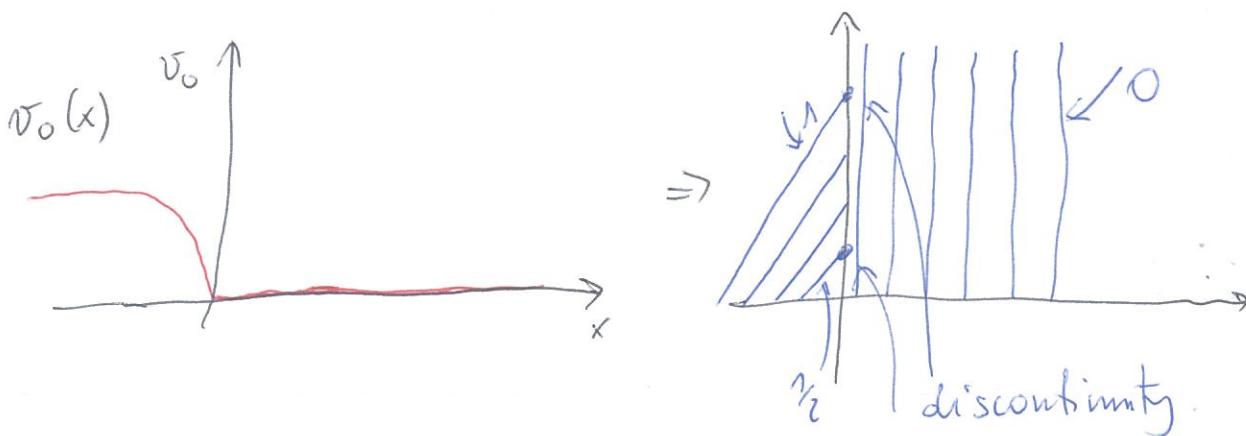
1. Hamilton-Jacobi equations

- Comes from Hamiltonian mechanics. In particular Jacobi was using it as a tool to integrate Hamilton's equations.
Has many applications in physics & control theory!

$$(*) \begin{cases} u_t + \frac{1}{2} |\nabla u|^2 = 0 \\ u(0, x) = u_0(x) \end{cases} \quad u: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$$

We cannot expect global-in-time regular solutions.

$$v := u_x \Rightarrow \begin{cases} v_t + v v_x = 0 \\ v(0, x) = v_0(x) \end{cases}$$



BUT! Concept of viscosity solutions!

Gives global-in-time solutions. Satisfying equation a.e.

They are unique. A solution to (*) can be represented by the formula! Hopf-Lax formula!

A bit degenerate H-Y equation.

Let $E_{ij}(x) = e^{-|x_i - x_j|}$, $x = (x_1, \dots, x_n)$.

$$\begin{cases} u_t + \frac{1}{2} \langle E(x) \nabla u, \nabla u \rangle = 0 \\ u(0, x) = u_0(x) \end{cases}$$

Coming from shallow water models.

We know that solutions (viscous) exist in any dimension.

In 1D we have representation formulas.

In 2D we know the optimal regularity (also in 1D).

u is time Lipschitz continuous and space $\frac{1}{2}$ Hölder continuous.

What about uniqueness in that class?

Hamilton-Jacobi equations and optimal control. ④

Hamilton-Jacobi-Bellman equations and Bellman's dynamic programming principle.

A process. $\begin{cases} \dot{x} = f(x, \alpha) \\ x(t) = x_0 \end{cases}, \alpha \in [-a, a] = A$ controls

Minimize (find a control α^* minimizing) a cost

$$J(x, \alpha) := \int_t^T l(x(s), \alpha(s)) ds + g(x(T)) \quad T - \text{grenz!}$$

Bellman's principle: $u(x, t) := \inf_{\alpha \in A} J(x, \alpha)$, then u satisfies

$$u(x, t) = \inf_{\alpha} \left\{ u(x(t+h), t+h) + \int_t^{t+h} l(x(s), \alpha(s)) ds \right\}.$$

Observation: $h \mapsto u(x(t+h), \alpha^*(t+h)) + \int_t^{t+h} l(x(s), \alpha^*(s)) ds$

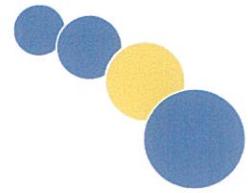
is constant in h provided α^* is the minimizing strategy.

Hence, $m_t + \nabla_u \cdot \dot{x} + l(x, \alpha^*) = 0 \quad \text{or}$

$$m_t + f(x, \alpha^*) \cdot \nabla_u + l(x, \alpha^*) = 0 ! \quad \text{H-J-B equation}$$

In other words: $m_t + \min_{\alpha \in A} \{ \nabla_u \cdot f(x, \alpha) + l(x, \alpha) \} = 0 . (*)$

And a feed-back control principle: solve $(*)$ and then
find $\alpha^* := \arg \min \{ \nabla_u \cdot f(x, \alpha) + l(x, \alpha) \} !$



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What sort of problems am I trying to solve using this method?

1. Economy consisting of evolution of capital, labour and resources. σ -rate of income reinvested in resources.
Our control (GDP).

Functional $J(k, N, \sigma) = \int_0^{+\infty} e^{-\delta t} U(k, N, \sigma) dt$, U -utility function.

2. Debt management by the National bank.

Our controls: rate of devaluation of the currency,
payment rate (the fraction of income/GDP that we use to pay the debt).

It is a bit game-theoretic problem. Comes from a report of Banco Espana 2015 (by Nuñez, Thomas).

3. Optimal control of the shape of magnets over the train in MAGLEV. Can't tell more!

Other PDEs that I'm interested in. Zikov's problem. (6)

$$\begin{cases} \Delta u + \operatorname{div}(bu) = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

$$\Omega \subset \mathbb{R}^n$$

Ω bounded,
noe boundary.

We look for a weak solution $u \in H_0^1(\Omega)$

If $b \in L^p$, $p \geq 2$ a unique solution exists.

$p < \frac{3}{2}$ - nonuniqueness! What about $\frac{3}{2} \leq p < 2$?

Our recent result (only a partial answer) involves quite complicated advanced techniques: convex integration and DiPerna-Lions commutative lemma.