Completely nonholonomic geometric distributions

(P. Mormul, IM UW)

The story starts with Frobenius theorem (Über das Pfaffsche Probleme, 1877): a subbundle $D \subset TM$, involutive, that is close wrt the Lie multiplication, $[D,D]=D$, gives rise to a foliation of M, $\mathcal{F}$, $D = T\mathcal{F}$ ← the vector spaces in D are tangent spaces to the leaves of $\mathcal{F}$

(Of course, the original wording of Frobenius was much different...)

First steps beyond the involutivness condition $[D,D]=D$:

Engel 1889, von Weber 1896, E. Cartan 1914, Goursat 1922.

After great expectations, they settled for a condition now called Goursat condition (should be called von Weber's):

$D \subset TM$ – a rank 2 subbundle,

$$\underset{\text{rk } 2}{D} \subset \underset{\text{rk } 3}{[D,D]} \subset \underset{\text{rk } 4}{[[D,D],[D,D]]} \subset \ldots\ldots \subset \underset{\text{rk} = \dim M}{TM}$$

– the tower of consecutive Lie squares of D, or else the derived flag of D growing regularly and slowly in ranks, always by 1. Every such a Goursat distribution is already completely nonholonomic: another tower, not necessarily of subbundles in M, just modules of vector fields tangent to M, the Lie flag $D_1 = D$, $D_{k+1} = [D, D_k]$, $k = 1, 2, 3, \ldots$, growing in linear dimensions (pointwise), possibly slowly and depending on a point, but until the maximal value $\dim M$.

Observation/exercise. D Goursat $\Rightarrow$ D completely nonholonomic, some Lie algebra needed, not much, Jacobi identity...

Just one example, $M = J^3(1,1)$, the jets of order 3 of functions $\mathbb{R} \to \mathbb{R}$, $t \mapsto x(t)$

$$\big(t, x(t), x'(t), x''(t), x'''(t)\big) = (t, x, x_1, x_2, x_3),$$

$$\left.\begin{aligned} x' = \frac{dx}{dt} &\longleftrightarrow dx - x'dt = 0 \\ x'' = \frac{dx'}{dt} &\longleftrightarrow dx' - x''dt = 0 \\ x''' = \frac{dx''}{dt} &\longleftrightarrow dx'' - x'''dt = 0 \end{aligned}\right\}$$

Pfaffian equations' description of the Cartan distribution $\mathcal{C} \subset TJ^3(1,1)$, $\operatorname{rk}\mathcal{C} = 2$.

Vector fields' description

$$\mathcal{C} = \operatorname{span}\begin{pmatrix} 1, & 0 \\ x', & 0 \\ x'', & 0 \\ x''', & 0 \\ 0, & 1 \end{pmatrix} \begin{matrix} \leftarrow \partial_t \\ \leftarrow \partial_x \\ \leftarrow \partial_{x'} \\ \leftarrow \partial_{x''} \\ \leftarrow \partial_{x'''} \end{matrix} = \operatorname{span}\begin{pmatrix} 1, & 0 \\ x_1, & 0 \\ x_2, & 0 \\ x_3, & 0 \\ 0, & 1 \end{pmatrix},$$

the derived flag of $\mathcal{C}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ 0, & 1, & 0 \\ 0 & 0 & 1 \end{pmatrix} \subset \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0, & 0, & 1, & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \subset \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0 & 0, & 0, & 0, & 1 \end{pmatrix}$$

$$\begin{matrix} \| & & \| & & \| \\ \underset{\operatorname{rk} 3}{[\mathcal{C},\mathcal{C}]} & \subset & \underset{\operatorname{rk} 4}{[[\mathcal{C},\mathcal{C}],[\mathcal{C},\mathcal{C}]]} & \subset & \underset{\operatorname{rk} 5}{TJ^3(1,1)} \end{matrix}$$

General feeling before 1978: (D Goursat) $\Rightarrow$ ($D \equiv \mathcal{C} \subset TJ^r(1,1)$, r = the length of the flag)

Not true — an earthquake in Paris 1978 / exclamation of prof. P. Lieberman /:

$$\mathcal{D} = \operatorname{span}\begin{pmatrix} x_3 & 0 \\ x_3 x_1 & 0 \\ x_3 x_2 & 0 \\ 1, & 0 \\ 0 & 1 \end{pmatrix},$$

Giaro–Kumpera–Ruiz, CRAS, 1978: $\mathcal{D}$ is Goursat, $\not\equiv \mathcal{C} \subset TJ^3(1,1)$.

Their original proof long & cumbersome. A much posterior proof in 1 line: the sequence of linear dim's of the Lie flag of $\mathcal{D}$ at $0 \in \mathbb{R}^5$ is $\underline{[2,3,4,4,5]}$ ← the small growth vector of $\mathcal{D}$ at $0 \in \mathbb{R}^5$ while the $\operatorname{sgrv}(\mathcal{C}) = \underline{[2,3,4,5]}$ at <u>every</u> point (the Cartan distr. on jets is homogeneous). This is the tip of an iceberg....

The sqrv's of Goursat distributions whatsoever

in dimension 5:

$$\left.\begin{array}{l}[2,3,4,5]\\ [2,3,4,4,5]\end{array}\right\} \# = 2 = F_{2\cdot 3-3}$$

in dimension 6:

$$\# = 5 = F_{2\cdot 4-3} = \left\{\begin{array}{l}[2,3,4,5,6]\\ [2,3,4,5,5,6]\\ [2,3,4,5,5,5,6]\\ [2,3,4,4,5,5,5,6]\\ [2,3,4,4,5,5,6]\end{array}\right.$$

All sqrv's of Goursat have been computed. In the underlying dimension $r+2$ (flag's length r) their $\# = F_{2r-3}$ (Fibonacci).

Over the past 20 years parallel theories are being developed, with the jumps in dimensions in the derived flag always by 2 (special 2-flags), or always by 3 (special 3-flags), etc.

Quite a lot of sqrv's of special 2-(3-,..) flags is by now known, but <u>not all</u>!! <u>Getting hold of all of them</u> is central in

the singularity theory of special multi-flags.

Differential geometry encounters here combinatorics, algebraic geometry....

In the year 2004 the computation of certain two sqrv's of special 2-flags in length 3: <u>[3,5,6,7,7,8,9]</u> and <u>[3,5,6,7,8,8,9]</u> allowed to discern two local geometric behaviours of 2-flags and to pinpoint <u>a serious mistake in a scientific paper</u>.