In the Department we work, in particular, with some graded geometrical objects and their applications in Mathematical Physics.

**Graded manifolds** are manifolds with an atlas whose local coordinates have associated weights (e.g. in $\mathbb{Z}_2$, $\mathbb{Z}$, or $\mathbb{N}$) and the transformations of coordinates respect the weight. There are many natural examples (e.g. supermanifolds, double vector bundles or higher tangent bundles) with applications in physics. This is a theory which is richer than just differential geometry.
Supermanifolds

The concept of a supermanifold comes from physics as a geometric tool to treat fermions and bosons on an equal footing. Local coordinates are therefore divided into the even and odd parity, so into traditional commuting variables $x^1, \ldots, x^n$ and anticommuting variables $y^1, \ldots, y^m$,

$$y^i \cdot y^j = -y^j \cdot y^i.$$ 

One can formalize this introducing a $\mathbb{Z}_2$ grading $\alpha$ in the algebra $C^\infty(\mathbb{R}^n)[y^1, \ldots, y^m]$ of ‘smooth functions’:

$$a \cdot b = (-1)^{\alpha(a)\alpha(b)} b \cdot a.$$ 

In this way we obtain the manifold $\mathbb{R}^{n|m}$.

One can develop a differential calculus on such objects, extending the traditional framework of smooth manifolds.
Supermanifolds–natural examples

Consider the traditional smooth manifold $M$ of dimension $n$ and its cotangent bundle $T^*M$. The space of differential forms

$$\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$$

is an algebra (with respect to the wedge product) canonically $\mathbb{N}$-graded, thus $\mathbb{Z}_2$-graded, the space $\Omega^k(M)$ being of degree $k$. Locally, $\Omega(M)$ is generated by coordinates $x^1, \ldots, x^n$ on $M$ and one-forms $y^i = dx^i$, $i = 1, \ldots, n$. The coordinates $x^i$ are commuting variables, the ‘coordinates’ $y^i$ are anticommuting

$$dy^i \wedge dx^j = - dx^j \wedge dx^i,$$

so we obtain a supermanifold denoted usually $\Pi T^*M$– the anti-tangent (odd tangent) bundle. Smooth functions on $\Pi T^*M$ form the algebra $\Omega(M)$.

Note that the Rham derivative $d : \Omega(M) \to \Omega(M)$ is an odd derivative of the algebra of smooth functions $\Omega(M)$ on $\Pi T^*M$, so a vector field on $\Pi T^*M$. 
Vector bundles as graded bundles

Another graded framework in differential geometry is a generalization of the concept of a vector bundle.

A vector bundle is a locally trivial fibration \( \tau : E \to M \) which, locally over \( U \subset M \), reads \( \tau^{-1}(U) \cong U \times \mathbb{R}^n \) and admits an atlas in which local trivializations transform linearly in fibers:

\[
U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n, \quad A(x) \in \text{GL}(n, \mathbb{R}).
\]

The latter property can also be expressed in the terms of the gradation in which base coordinates \( x \) have degrees 0, and ‘linear coordinates’ \( y \) have degree 1. Linearity of changes of coordinates is now equivalent to the fact that changes of coordinates respect the degrees.

A natural generalization is to associate with coordinates \( (y^i) \) in \( \mathbb{R}^n \) weights (degrees) \( w_i = 1, \ldots, d \) and require that changes of coordinates respect the weights. The obtained objects, graded bundles of degree \( d \), are no longer linear: if we take variables \( x, y, z \) of weights 0, 1, 2, then the map \( (x, y, z) \mapsto (x, y, z + y^2) \) preserves the weight but is not linear.
Consider the second-order tangent bundle $T^2M$, i.e. the bundle of second jets of smooth curves maps $(\mathbb{R}, 0) \to M$. ($TM$ is a vector bundle.) Writing curves in local coordinates $(x^A)$ on $M$:

$$x^A(t) = x^A(0) + \dot{x}^A(0)t + \ddot{x}^A(0)\frac{t^2}{2} + o(t^2),$$

we get local coordinates $(x^A, \dot{x}^B, \ddot{x}^C)$ on $T^2M$, which transform

\[
egin{align*}
\dot{x}'^A &= x'^A(x), \\
\ddot{x}'^A &= \frac{\partial x'^A}{\partial x^B}(x) \ddot{x}^B, \\
\dddot{x}'^A &= \frac{\partial x'^A}{\partial x^B}(x) \dddot{x}^B + \frac{\partial^2 x'^A}{\partial x^B \partial x^C}(x) \ddot{x}^B \dddot{x}^C.
\end{align*}
\]

This shows that associating with $(x^A, \dot{x}^B, \ddot{x}^C)$ the weights $0, 1, 2$, respectively, will give us a graded bundle structure of degree $2$ on $T^2M$. Due to the quadratic terms above, this is not a vector bundle! All this can be generalised to higher tangent bundles $T^kM$. 