GRADED MANIFOLDS IN GEOMETRY AND PHYSICS

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In the Department we work, in particular, with some graded geometrical objects and their applications in Mathematical Physics.

Graded manifolds are manifolds with an atlas whose local coordinates have associated weights (e.g. in $\mathbb{Z}_2$, $\mathbb{Z}$, or $\mathbb{N}$) and the transformations of coordinates respect the weight. There are many natural examples (e.g. supermanifolds, double vector bundles or higher tangent bundles) with applications in physics. This is a theory which is reacher than just differential geometry.

Supermanifolds

The concept of a supermanifold comes from physics as a geometric tool to treat fermions and bosons on an equal footing.
Local coordinates are therefore divided into the even and odd parity, so into traditional commuting variables $x^1, \ldots, x^n$ and anticommuting variables $y^1, \ldots, y^m$,

$$y^i \cdot y^j = -y^j \cdot y^i \,.$$

One can formalize this introducing a $\mathbb{Z}_2$ grading α in the algebra $C^\infty(\mathbb{R}^n)[y^1, \ldots, y^m]$ of 'smooth functions':

$$a \cdot b = (-1)^{\alpha(a)\alpha(b)} b \cdot a \,.$$

In this way we obtain the manifold $\mathbb{R}^{n|m}$.

One can develop a differential calculus on such objects, extending the traditional framework of smooth manifolds.

Supermanifolds–natural examples

Consider the traditional smooth manifold M of dimension n and its cotangent bundle T^*M. The space of differential forms

$$\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$$

is an algebra (with respect to the wedge product) canonically $\mathbb{N}$-graded, thus $\mathbb{Z}_2$-graded, the space $\Omega^k(M)$ being of degree k. Locally, $\Omega(M)$ is generated by coordinates $x^1, \ldots, x^n$ on M and one-forms $y^i = \mathrm{d}x^i$, $i = 1, \ldots, n$. The coordinates x^i are commuting variables, the 'coordinates' y^i are anticommuting

$$\mathrm{d}x^i \wedge \mathrm{d}x^j = -\,\mathrm{d}x^j \wedge \mathrm{d}x^i\,,$$

so we obtain a supermanifold denoted usually $\Pi\mathsf{T}M$– the anti-tangent (odd tangent) bundle. Smooth functions on $\Pi\mathsf{T}M$ form the algebra $\Omega(M)$.

Note that the Rham derivative $\mathrm{d}\colon \Omega(M) \to \Omega(M)$ is an odd derivative of the algebra of smooth functions $\Omega(M)$ on $\Pi\mathsf{T}M$, so a vector field on $\Pi\mathsf{T}M$.

Vector bundles as graded bundles

Another graded framework in differential geometry is a generalization of the concept of a vector bundle.

A vector bundle is a locally trivial fibration $\tau : E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibers

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n , \quad A(x) \in \mathrm{GL}(n, \mathbb{R}) .$$

The latter property can also be expressed in the terms of the gradation in which base coordinates x have degrees 0, and 'linear coordinates' y have degree 1. Linearity of changes of coordinates is now equivalent to the fact that changes of coordinates respect the degrees.

A natural generalization is to associate with coordinates (y^i) in $\mathbb{R}^n$ weights (degrees) $w_i = 1, \ldots, d$ and require that changes of coordinates respect the weights. The obtained objects, graded bundles of degree d, are no longer linear: if we take variables x, y, z of weights $0, 1, 2$, then the map $(x, y, z) \mapsto (x, y, z + y^2)$ preserves the weight but is not linear.

Natural examples of graded bundles

Consider the second-order tangent bundle T^2M, i.e. the bundle of second jets of smooth curves maps $(\mathbb{R},0)\to M$. ($\mathsf{T}M$ is a vector bundle.)
Writing curves in local coordinates (x^A) on M:

$$x^A(t)=x^A(0)+\dot{x}^A(0)t+\ddot{x}^A(0)\frac{t^2}{2}+o(t^2)\,,$$

we get local coordinates $(x^A,\dot{x}^B,\ddot{x}^C)$ on T^2M, which transform

$$\begin{aligned}
x'^A &= x'^A(x)\,,\\
\dot{x}'^A &= \frac{\partial x'^A}{\partial x^B}(x)\,\dot{x}^B\,,\\
\ddot{x}'^A &= \frac{\partial x'^A}{\partial x^B}(x)\,\ddot{x}^B+\frac{\partial^2 x'^A}{\partial x^B\partial x^C}(x)\,\dot{x}^B\dot{x}^C\,.
\end{aligned}$$

This shows that associating with $(x^A,\dot{x}^B,\ddot{x}^C)$ the weights $0,1,2$, respectively, will give us a graded bundle structure of degree 2 on T^2M.
Due to the quadratic terms above, this is not a vector bundle!
All this can be generalised to higher tangent bundles T^kM.