Representations of Dynamical Systems

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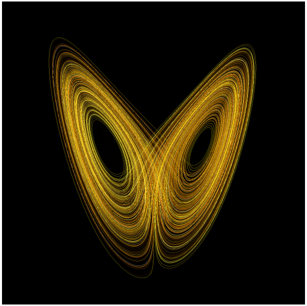
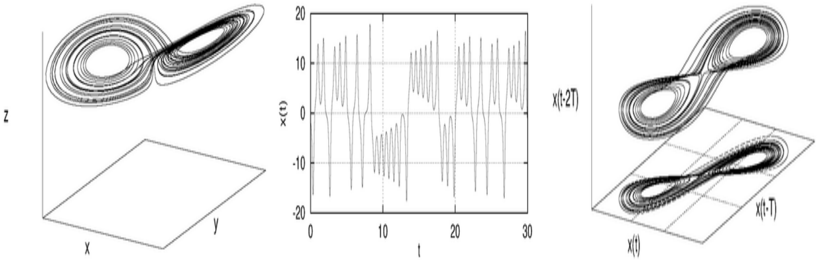
The Lorenz attractor



$$\begin{cases} \dot{x} &= 10(y - x) \\ \dot{y} &= 28x - y - xz \\ \dot{z} &= xy - \frac{8}{3}z \end{cases}$$

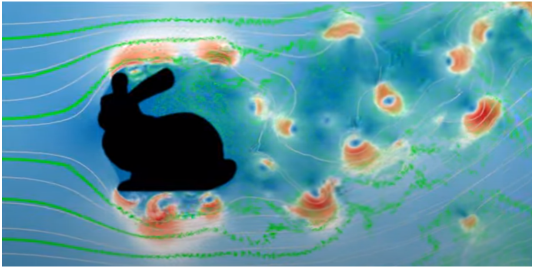
Experimental reconstruction of a system



$$t \mapsto \big(x(t), x(t-T), x(t-2T)\big)$$

Source: Timothy D. Sauer (2006) *Attractor reconstruction*. Scholarpedia, 1(10):1727.

2D Navier-Stokes flow



Source: G.H. Keetels, H.J.H. Clercx, G.J.F. van Heijst, *Fourier spectral solver for the incompressible Navier-Stokes equations with volume penalization*, Proceedings of the 7th International Conference on Computational Science, Beijing, China, 2007; 898-905.

d-cubical shifts

Let $d \in \mathbb{N}$. Denote:

$$X = ([0,1]^d)^{\mathbb{Z}}$$

with product topology. Let $T : X \to$ be the shift homeomorphism:

$$(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$$

$$\downarrow$$

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$(([0,1]^d)^{\mathbb{Z}}, \mathit{shift})$ is referred to as the full topological shift on the alphabet $[0,1]^d$ or simply as the d-cubical shift.

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Embedding dimension

- For any compact metrizable space X there is a (topological) embedding into the Hilbert cube: $\phi : X \hookrightarrow [0,1]^{\mathbb{N}}$.
- Let (X,T) be a t.d.s. There is a (dynamical) embedding by the orbit-map:

$$\begin{aligned} \Phi : (X,T) &\hookrightarrow (([0,1]^{\mathbb{N}})^{\mathbb{Z}}, \mathrm{shift}) \\ x &\mapsto (\phi(T^k x))_{k\in\mathbb{Z}} \end{aligned}$$

- Under which conditions is there an embedding into the d-cubical shift:

$$(X,T) \hookrightarrow (([0,1]^{d})^{\mathbb{Z}}, \mathrm{shift}) \; (d \in \mathbb{N})?$$

- Define the embedding dimension:

$$\mathrm{edim}(X,T) = \min\{d \in \mathbb{N} \cup \{\infty\} \,|\, (X,T) \hookrightarrow (([0,1]^{d})^{\mathbb{Z}}, \mathrm{shift})\}$$

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The Lindenstrauss-Tsukamoto Conjecture

Conjecture (Lindenstrauss-Tsukamoto, 2014)

Let $d \in \mathbb{N}$ be such that

$$\operatorname{perdim}(X, T) < \frac{d}{2}$$

$$\operatorname{mdim}(X, T) < \frac{d}{2}$$

then $(X, T) \hookrightarrow (([0, 1]^d)^{\mathbb{Z}}, \text{shift})$.

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Mean dimension (metric definition)

- Let $f : X \to Y$ be a continuous map and $\varepsilon > 0$. The map f is called an ε-embedding if $\operatorname{diam} f^{-1}(y) < \varepsilon$ for all $y \in Y$.
- Let $\operatorname{widim}_\varepsilon(X, d)$ be the minimum integer $n \geq 0$ such that there exist an n-dimensional simplicial complex P and an ε-embedding $f : X \to P$.
- $\dim(X) = \lim_{\epsilon \to 0} \operatorname{widim}_\epsilon(X, d)$
- $d_n(x, y) = \max_{0 \leq i \leq n-1} d(T^i x, T^i y)$
- (Gromov) $\boxed{\operatorname{mdim}(X, T) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\operatorname{widim}_\epsilon(X, d_n)}{n}}$

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Periodic dimension obstruction for embedding

Let $P_n = \{x \in X| \ \exists 1 \leq m \leq n \ T^m x = x\}$, be the set of periodic points of period $\leq n$ of X. Define:

$$\text{perdim}(X,T) = \Big(\frac{\dim(P_1)}{1}, \frac{\dim(P_2)}{2}, \ldots\Big)$$

Notation: $d \geq \text{perdim}(X,T)$ if for every $m \in \mathbb{N}$, $d \geq \text{perdim}(X,T)|_m$

- $\text{perdim}(([0,1]^d)^{\mathbb{Z}}, \text{shift}) = (d, d, \ldots)$
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Thank you!

Dziękuję bardzo!