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Wydział Matematyki, Informatyki i Mechaniki
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OPERATOR ALGEBRAS THAT ONE CAN SEE — A FORETASTE OF NONCOMMUTATIVE TOPOLOGY

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What is a compact quantum space?

Theorem (Gelfand-Naimark)

Every C-algebra is a complex algebra of continuous (i.e. bounded) linear operators on a complex Hilbert space that is:*

1. *a topologically closed set in the norm topology of operators,*
2. *closed under the operation of taking adjoints of operators.*

Theorem (Gelfand-Naimark)

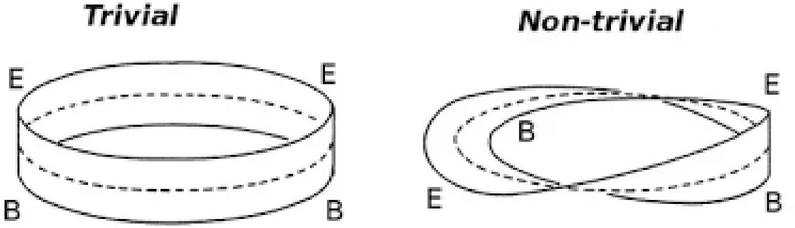
Every commutative C-algebra is naturally isomorphic to the algebra of all continuous complex-valued vanishing-at-infinity functions on a locally compact Hausdorff space.*

Theorem

The category CH of compact Hausdorff spaces and the category CC of unital commutative C*-algebras are anti-equivalent via the contravariant functor*

$$C\colon \mathit{CH} \ni X \longmapsto C(X) \in \mathit{CC}^*.$$

What is a noncommutative vector bundle?



Theorem (Serre–Swan)

The category CVB of complex vector bundles over compact Hausdorff spaces and the category FGP of finitely generated projective modules over unital commutative C-algebras are anti-equivalent via the contravariant functor*

$$C\colon \mathit{CVB} \ni (E \to X) \longmapsto C(E \to X) \in \mathit{FGP}.$$

Graphs and their algebras

Definition

A graph (oriented graph, quiver) $E := (E^0, E^1, s, t)$ is a quadruple consisting of the set of vertices E^0, the set of edges (arrows) E^1, and the source and target (range) maps $s, t\colon E^1 \to E^0$ assigning to each edge its source and target vertex respectively.

Definition

Let E be a graph. The universal C*-algebra $C^*(E)$ of the graph E is generated by mutually orthogonal projections $\{P_v \mid v \in E^0\}$ and partial isometries $\{S_x \mid x \in E^1\}$ satisfying the relations:

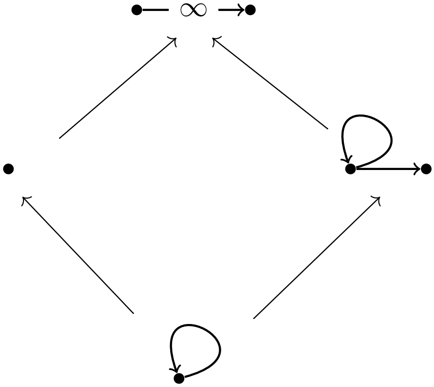
1. $\forall\, x\,, y \in E^1 :\ S_x^* S_y = \delta_{xy} P_{t(x)}$,
2. $\forall\, v \in E^0$ such that the preimage $s^{-1}(v)$ is not empty and finite: $\sum_{x \in s^{-1}(v)} S_x S_x^* = P_v$,
3. $\forall\, x \in E^1 :\ S_x S_x^* \leqslant P_{s(x)}$.

Examples of pushouts

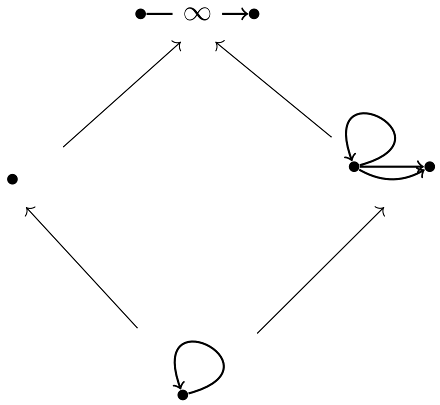
$$
\begin{tikzcd}
& S^2 & \\
\text{Point} \arrow[ur] & & D \arrow[ul] \\
& S^1 \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& S^2 & \\
\text{Point} \arrow[ur] & & \mathbb{R}\mathrm{P}^2 \arrow[ul] \\
& S^1 \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& WP^1(1,2) & \\
\text{Point} \arrow[ur] & & S^2 \arrow[ul] \\
& S^1 \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{C}\mathrm{P}^n & \\
\mathbb{C}\mathrm{P}^{n-1} \arrow[ur] & & B^{2n} \arrow[ul] \\
& S^{2n-1} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

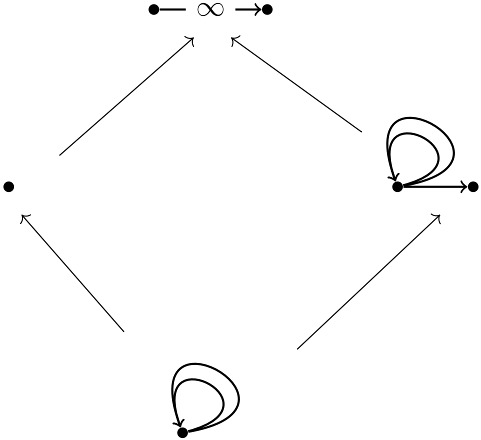
Quantum $\mathbb{C}P^1$



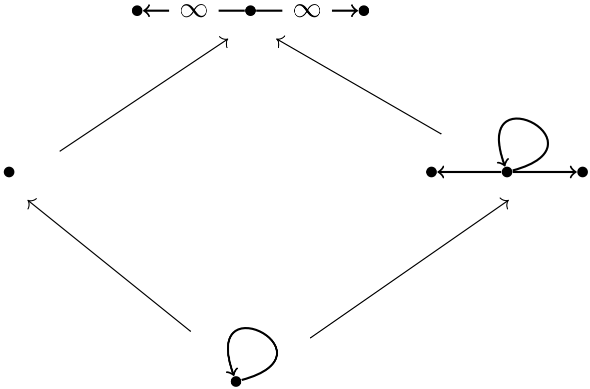
Quantum $\mathbb{C}P^1$ revisited



A quantum bonus



Quantum weighted complex projective line



Quantum $\mathbb{C}P^2$