# A probabilistic approach to PDEs

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For the last two decades, the interest in partial differential equations (PDEs) with non-local operators has grown rapidly. At many prestigious research centers, some groups were formed that would carry out research on this type of equations and problems related. Let us mention here the papers of

- Ambrosio [1], Barrios, Figalli and Valdinoci [2], Bass and Kassmann [3],
- Berestycki, Coville and Vo [4], Cabré and Sire [5], Caffarelli, Silvestre et al. [1, 2],
- Caffarelli, Dipierro and Valdinoci [3], Chen and Véron [4], Kuusi, Mingione and Sire [6],
- Mawhin and Molica Bisci [1], Priola and Zabczyk [2], Ros-Oton and Serra [3],
- Servadei and Valdinoci [4, 5], Vázquez et al. [5, 6, 7].

The basic difference between local and non-local operator is that the evaluation of non-local operator on a function $u$ at point $x \in \mathbb{R}^d$ depends on values of $u$ on the whole $\mathbb{R}^d$ and not, as in the case of local operator, on values of $u$ on an arbitrary small neighborhood of $x$.

The fundamental non-local operator is the fractional Laplacian $\Delta^\alpha := -(-\Delta)^\alpha$ with $\alpha \in (0,1)$.

$$\Delta^\alpha u(x) := \int_{\mathbb{R}^d} \Big( u(x+y) - u(x) - \frac{y \cdot \nabla u(x)}{1+|y|^2} \Big) \frac{c_\alpha}{|y|^{d+2\alpha}}\, dy, \quad x \in \mathbb{R}^d. \quad (1)$$

The above integral is well-defined at least for functions in $C_c^\infty(\mathbb{R}^d)$.
In this special case, we may also write equivalently that

$$\Delta^\alpha u(x) = c_\alpha \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} \frac{u(x+y) - u(x)}{|y|^{d+2\alpha}}\, dy. \quad (2)$$

These types of operators have been investigated in quantum physics for a long time. Recall here relativistic Schrödinger operator

$$-A = \sqrt{-\Delta + m^2} - m.$$

However, the recent surge in interest in this topic has its roots elsewhere. The numerous scientific publications revealed that in the large part of physical, biological, chemical, and mathematical finance models the substitution of classical operators by non-local operators in related PDEs leads to the solutions that better describe the phenomena.

# Lévy type operators

Let $D$ be an open subset of $\mathbb{R}^d$. Consider the following integro-differential operator

$$Au(x) := Lu(x) + Su(x), \quad x \in D,$$

where

$$Lu(x) = \frac{1}{2}\mathrm{Tr}(\mathbf{Q}(x)\nabla^2 u(x)) + b(x) \cdot \nabla u(x), \quad x \in D$$

and

$$Su(x) = \int_{\mathbb{R}^d} \left( u(x+y) - u(x) - \frac{y \cdot \nabla u(x)}{1+|y|^2} \right) N(x, dy), \quad x \in D$$

for $u \in C_b^2(D)$.
For the fractional Laplacian, we have

$$N(x, dy) = \frac{c_\alpha}{|y|^{d+2\alpha}}\, dy.$$

# Coefficients of the operator

The coefficients of the operators satisfy:

- $q_{i,j}, b_j \in \mathcal{B}_b(D)$, $i,j = 1, \dots, d$, $\mathbf{Q}(x) = [q_{i,j}(x)]_{i,j=1}^d$ is non-negative definite for any $x \in D$, i.e.

$$\sum_{i,j=1}^{d} q_{i,j}(x)\xi_i\xi_j \geqslant 0, \quad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d,$$

- $N(x, dy)$ (*Lévy kernel*) is a $\sigma$-finite positive Borel measure on $\mathbb{R}^d \setminus \{0\}$ for fixed $x \in D$, and for any $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ the function

$$x \mapsto N_B(x) := \int_B \min\{1, |y|^2\} N(x, dy)$$

  is measurable.

$$N_* := \sup_{x \in D} N_{\mathbb{R}^d}(x) < +\infty.$$

To give a flavor of the differences and problems, let us consider one of the most classical problem in PDEs, i.e. the Dirichlet problem on a bounded domain $D \subset \mathbb{R}^d$.

Here, even on the conceptual level appears a difficulty.

- As in the case of the classical Laplacian we require that

$$-\Delta^\alpha u(x) = 0, \quad x \in D \qquad (u(x) = \varphi(x),\, x \in \partial D?).$$

  However, $u$ has to be defined on the whole $\mathbb{R}^d$ as it is required in the definition of the fractional Laplacian. Thus, we need to determine the values of $u$ on $D^c$. Therefore, in case of the fractional Laplacian it seems natural to reformulate the Dirichlet boundary condition into an *exterior condition*, and require that $u(x) = \varphi(x)$, $x \in D^c$ for given $\varphi \in \mathcal{B}(D^c)$.

- The next far-reaching consequence of the non-local character of $\Delta^\alpha$ is that strong maximum principle holds for **any bounded open set** (connectedness is not required contrary to the classical Laplacian).

- The next significant difference is that solutions of fractional Poisson equations are non-regular up to the boundary even if all data are smooth e.g. solution of $-\Delta^\alpha u = 1$ on $B(0,1)$ with zero exterior condition is not even Lipschitz up to the boundary (see [3]). Thus, it is not clear how to define other boundary problems like Neumann problem for the fractional Laplacian or the Hopf lemma.
- Finally, let us mention the difference between classical and the fractional Laplacian which is crucial for variational inequalities approach to PDEs. Namely, if $\mathcal{E}$ is the Dirichlet form generated by $\Delta^\alpha$ on $H^\alpha(\mathbb{R}^d)$, i.e.

  $$\mathcal{E}(u,v) = \int_{\mathbb{R}^d} (-\Delta^\alpha u, v), \quad u \in D(\Delta^\alpha), v \in H^\alpha(\mathbb{R}^d),$$

  then it is not true that $\mathcal{E}(u \wedge 1, u) = \mathcal{E}(u \wedge 1, u \wedge 1)$ for $u \in H^\alpha(\mathbb{R}^d)$ (as it holds in the case of classical Laplacian). Therefore, many truncation methods needed for equations with less regular data are not applicable (at least directly) to the case of non-local PDEs.

- In the last decade, a great effort has been made to develop the theory of PDEs with the fractional Laplacian. A lot of papers have been written on this topic, and highly interesting methods have been developed.
- However, most of these techniques work only for the fractional Laplacian or some of its small variation, and every time there is a need to change the driving operator in the problem, the whole work has to be repeated.
- Recall here the commonly used technique, considered for the first time in the context of PDEs by Caffarelli and Silvestre in [2], which allows one to reformulate some PDEs and related problems with the fractional Laplacian into the problems with degenerate local operator on an extended domain. In practice, application of this method is limited to the fractional Laplacian.

# Probabilistic approach

Under some assumptions on the operator $A$ there exists a family $(P_x)$ of probability measure on the Skorokhod space $\mathcal{D}$ (trajectories) of possible trajectories of particle under dynamics given by the operator $A$, i.e.

$$T_t f(x) = \int_{\mathcal{D}} f(\omega(t))\, dP_x(\omega), \quad x \in \mathbb{R}^d.$$

We let $X_t(\omega) := \omega(t)$.
Let $\tau_D := \inf\{t > 0 :\ X_t \notin D\}$.

## Definition (Probabilistic solutions)

Let $u \in \mathcal{B}_b(\mathbb{R}^d)$ be a solution to

$$-Av(x) = g(x), \quad x \in D. \tag{3}$$

Then

$$u(x) = \mathbb{E}_x u(X_{\tau_D}) + \mathbb{E}_x \int_0^{\tau_D} g(X_r)\, dr, \quad x \in D.$$

# Goals of the project

- Strong maximum principle for nonlinear Schrödinger equations;
- Existence of solutions to Schrödinger equations with singular potential;
- Hopf's lemma for Schrödinger operators;
- Kato's inequality;
- Removable and isolated singularities.

As to the last point, we would like to focus on the following problem of *isolated singularities* considered by P. Lions. Suppose that $u$ is a positive classical solution to

$$-\Delta u = u^p \quad \text{in} \quad D \setminus \{0\}, \qquad u = 0 \quad \text{on} \quad \partial D. \tag{4}$$

Here $p > 1$ and $0 \in D$. The problem is to determine behavior of the function $u$ on the whole $D$. This problem was recently considered by Chen and Quaas for the fractional Laplacian.

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