Conformal geometry and differential equations

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22.05.2020

Overview

- This project is a part of a **NCN grant** entitled “Geometry and integrable systems”.
- Main objective: exploit interactions between **geometry** and **differential equations**.

General approach

$$\text{PDE} \qquad \longleftrightarrow \qquad \text{geometric structure}$$

- construct a correspondence between equations (or solutions to a given equation) and geometric objects
- describe properties of PDEs through geometry (e.g. integrability, conservation laws, linearization etc)
- construct solutions to equations through geometric methods, and conversely construct interesting geometries from well known solutions of classical equations
- examples: twistor theory (Penrose), Pfaffian systems (Cartan, Kahler), 3rd order ODEs and conformal geometry (Chern)

Today, I'll concentrate on a particular case on the geometrical side: the conformal geometry.

Conformal geometry

A **conformal structure** on a manifold M is a class of equivalent (pseudo)-Riemannian metrics $[g]_\sim$ where

$$g \sim \hat{g} \qquad \text{iff} \qquad g = \varphi \hat{g}$$

for some positive function $\varphi \colon M \to \mathbb{R}_+$.
It follows that a distance on M is not defined, however one can measure angles between vectors

$$\cos \angle(X, Y) = \frac{g(X, Y)}{\|X\|_g \, \|Y\|_g}$$

In the Lorenzian signature the **light cones** (or **null cones**)

$$C = \{X \in TM \,:\, \|X\|_g = 0\}$$

are well defined by $[g]_\sim$ (it is crucial in general relativity).

PDEs and conformal geometry

Consider a 2nd order PDE for a function $u = u(x, y, z)$.

$$E(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, u_{xy}, u_{xz}, u_{yz}) = 0$$

e.g. the dispersionless Hirota equation

$$au_{xy}u_z + bu_{yz}u_x - (a+b)u_{xz}u_y = 0.$$

The **symbol** of a linear PDE is a homogeneous polynomial function on the co-tangent bundle T^*M given up to a **factor**. For a nonlinear equation for any solution there is a **symbol of the linearized** equation along the solution.

For the equation E the symbol is a quadratic function on $T^*\mathbb{R}^3$. The inverse of the corresponding bi-linear forms gives a **conformal structure** (depending on a given solution u).

Can we study analytical properties of E through the geometry?

Integrable PDEs

Theorem (Ferapontov-Kruglikov, J. Diff. Geom., 2014)

E is integrable iff the conformal structure for any solution u admits a connection ∇ that satisfies the Einstein-Weyl equation.

In above: an **Einstein-Weyl structure** is a conformal class $[g]_\sim$ with a connection ∇ satisfying the conformal **Einstein equation**

$$Ric(\nabla) = \frac{1}{n} R_g(\nabla) \cdot g$$

Applications: take a well know integrable system and construct solutions to the Einstein-Weyl equation explicitly. E.g. for the Hirota equation

$$g = \frac{w_x}{w_y w_z} dx^2 + \frac{b^2 w_y}{a^2 w_x w_z} dy^2 + \frac{(a+b)^2 w_z}{a^2 w_x w_y} dz^2$$
$$- 2\frac{b}{a w_z} dxdy + 2\frac{a+b}{a w_y} dxdz + 2\frac{b(a+b)}{a^2 w_x} dydz.$$

Problems

- Generalize this to other classes of PDEs and geometries (partially done e.g. for $GL(2)$-geometry and so-called Cayley structures).
- Find the most general PDEs whose solutions cover all special geometries of a given type (e.g. all Einstein-Weyl structures in the previous setting). Solve this equation, e.g. via twistor methods, and get normal forms for geometric structures.
- A particular equation to consider: the Camassa-Holm equation.

Other topics in the grant:

- Geometric control theory and sub-Riemannian geometry.
- Geometry of ODEs (another link to conformal geometry).

Grant

- The project is a part of the grant NCN no. 2019/34/E/ST1/00188.
- The grant includes two PhD positions.
- Additional scholarship in the grant: 2000PLN monthly.

Thank you for your attention!

In case of questions please contact me at:

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