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New homological methods in lattice theory, convex geometry and information theory

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Generalized 3rd Hilbert's Problem

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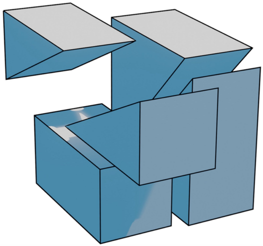
Problem

Are all convex polytopes of the same dimension and volume scissors equivalent*?*



David Hilbert $\sim$ 1900

Dehn invariant and scissors equivalence

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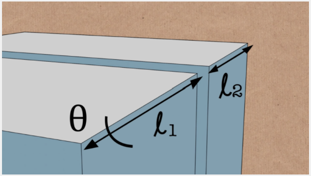
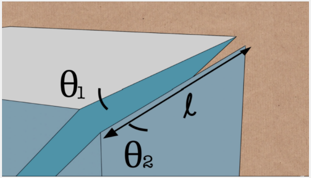
$\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/2\pi\mathbb{Z})$

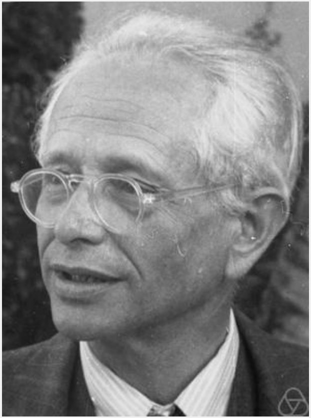


$$(\ell_1 + \ell_2) \otimes \theta = \ell_1 \otimes \theta + \ell_2 \otimes \theta$$



$$\ell \otimes (\theta_1 + \theta_2) = \ell \otimes \theta_1 + \ell \otimes \theta_2$$

Max Dehn $\sim$ 1945



Dehn invariant and Milnor K-theory

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Theorem. [Matsumoto]

$$K_2^M(F) = \frac{F^\times \otimes_{\mathbb{Z}} F^\times}{(a \otimes (1-a) \mid a \neq 0, 1)}.$$

For a field $F \subset \mathbb{R}$ related to the Dehn invariant by

1. exact sequences,
2. the *scissors group*,
3. group homology of SO_3,
4. and *polylogarithms*.

John Milnor ~ 1970

Scissors group and Information Theory

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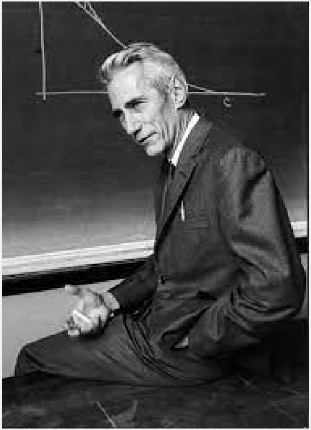
Shannon entropy ($F = \mathbb{R}$)

The relations defining *infinitesimal scissors group* of Cathelineau $\sim$ Fundamental Equation of Information Theory.

$$f(x) + (1-x)f(\frac{y}{1-x}) \\ = f(y) + (1-y)f(\frac{x}{1-y}),$$

for $x, y \in [0,1)$, $x + y \leq 1$, leading to the Shannon entropy.

Claude Shannon $\sim$ 1980



Hochschild cohomology for abstract convexity

The idea of convexity as an abstract algebraic structure goes back to Stone.

Theorem (*Hochschild Cohomology*, TM '21)

The abstract convex structure admits a theory of Hochschild extensions classified by a second cohomology of a canonical complex.

Theorem (*Shannon Cohomology Class*, TM '21)

There exists a canonical complex $A^\bullet$ built from the convex structure of a simplex such that $H^2(A^\bullet) \cong \mathbb{R}$ is generated by the Shannon entropy.

Entropy on convex polytopes

Definition (*Entropy*, TM '21)

Using the fact that every finite convex polytope is a canonical affine image of a simplex, allows one to use the MaxEnt principle of Jaynes to define the canonical entropy function for any finite convex polytope.

Information metric of convex polytopes

Theorem (*Information Metric*, TM '21)

The canonical entropy defines an affinely invariant Riemannian metric on a polytope generalizing the Fisher information metric of a simplex.

Example (Friedrich '91)

The simplex with the information metric is a part of a sphere.

Example (TM '21)

The square with our information metric is a part of a flat torus.

Remark

The 3rd Hilbert problem makes sense for our information metric.

Replication as minimizing entropy flow

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Theorem (*Information Flow*, TM '21)

The inverse gradient flow of the Shannon entropy with respect to the Fisher information metric on a simplex coincides with the smoothing of the process of simple replication.

Remark

This makes sense for any finite convex polytope.

Remark

The polytope case can model the replication process of coevolving replicators.

List of problems

1. 3rd Hilbert's problem for finite convex polytopes with the generalized information metric.
2. Computation of Hochschild cohomology of finite convex polytopes.
3. Studying information geometry of finite convex polytopes.
4. Studying information flow on finite convex polytopes.
5. Applications to
    a. complexity of polytopes and problems of linear programming,
    b. the Kolmogoroff complexity or topological complexity of networks and dynamical systems.
    c. the fundamental problems related to replicators in abiogenesis or evolutionary and molecular biology,
    d. Bayesian belief revision in medical diagnostics and epidemiological models.