UNIVERSITY OF WARSAW

Quantum coverings of manifolds and low dimensional topology

Tomasz Maszczyk
t.maszczyk@uw.edu.pl

May 23, 2021

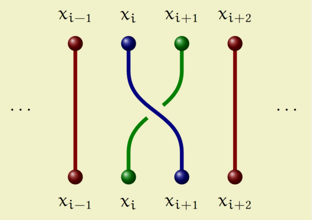
Table of contents

Braid diagrams on the plane

UNIVERSITY OF WARSAW

Artin Braid Group

Isotopy classes of braids in $\mathbb{R}^3$ form a group with explicit generators and relations.



Braids can be projected onto their braid diagrams in $\mathbb{R}^2$.

Emil Artin $\sim$ 1940

Braid diagrams on Riemann surfaces

UNIVERSITY OF WARSAW

Tubular Braids

Similarly, homotopy classes of braids in a tubular neighborhood of a Riemann surface $X \subset \mathbb{R}^3$ can be projected into X.

Theorem (Theodoro De Lima '19)

Every n-strand generalized string link on a surface X is link-homotopic to a braid.

Juliana Roberta Theodoro De Lima ∼ 2021

Covering spaces and monodromy

Classical Monodromy

Monodromy is a parallel transport along homotopy class of a path.

In the monodromy correspondence, Galois covers of X correspond to quotient groups of $\pi_1(X, x_0)$.

Problem

While the monodromy of Galois covers can detect homotopy classes of strands in X, it does not see the braiding between different strands.

One needs a better device than the classical monodromy to detect braids on X, i.e. the homotopy classes of their strands and the braiding.

Hopf-Galois theory

Hopf-Galois extensions

An H-comodule algebra A over a Hopf algebra H

$$A \to A \otimes H,$$
$$a \mapsto a_{(0)} \otimes a_{(1)},$$

with the subalgebra of invariants

$$B = \{b \in A \mid b_{(0)} \otimes b_{(1)} = b \otimes 1\}$$

and bijective canonical map

$$A \otimes_B A \to A \otimes H,$$
$$a \otimes_B a' \mapsto aa'_{(0)} \otimes a'_{(1)}.$$

Mitsuhiro Takeuchi $\sim$ 1980

Quantum Galois covering spaces

UNIVERSITY OF WARSAW

Definition (TM '19)

An H-Galois extension A of $B = C^\infty(M)$ with a finite cosemisimple coribbon Hopf algebra H and a parallel transport along smooth homotopy classes of braid diagrams on a manifold M is called quantum Galois covering of M.

Theorem (*Quantum Monodromy*, Bigdeli–TM '21)

Provided some class in the 2nd Gerstenhaber-Schack cohomology of the Hopf algebra H vanishes, every quantum H-Galois covering of a smooth manifold M admits a parallel transport along paths in M.

There is a quantum analog of the Maurer-Cartan equation for that parallel transport making it depend only on the smooth homotopy class of a braid diagram on M.

Quantum fundamental group

Solutions to the classical Maurer-Cartan equation are representations of the fundamental group in a Lie group G.

Solutions to the quantum Maurer-Cartan equation should be regarded as representations of the quantum fundamental group in a quantum group H.

A candidate for an appropriate notion of the quantum fundamental group and its quantum representation variety was proposed by Habiro in 2011.

List of problems

1. Study the moduli space of solutions to the quantum Maurer-Cartan equation.
2. Compare this moduli space with Habiro's construction.
3. Construct an analog of the Chern-Simons theory for the quantum Maurer-Cartan equation.