Completely nonholonomic geometric distributions
(P. Mormul, IM UW)

The story starts with Frobenius theorem (Über das Pfaffsche Problem, 1877): a subbundle $D \subset TM$, involutive, that is close with the Lie multiplication, $[D, D] = D$, gives rise to a foliation of $M$, $\mathcal{F}$, $D = TF$ - the vector spaces in $D$ are tangent spaces to the leaves of $\mathcal{F}$.

(Of course, the original wording of Frobenius was much different...)

First steps beyond the involutiveness condition $[D, D] = D$:
Engel 1889, von Weber 1896, E. Cartan 1914, Goursat 1922.

After great expectations, they settled for a condition now called Goursat condition (should be called von Weber's):

$D \subset TM$ - a rank 2 subbundle,

$$D \subset [D, D] \subset \ldots \subset [D, D], [D, D] \subset \ldots \subset TM$$

$rk 2 \quad rk 3 \quad rk 4 \quad \ldots \quad rk = \dim M$

- the tower of consecutive Lie squares of $D$, or else
the derived flag of $D$ growing regularly and slowly in ranks, always by 1. Every such a Goursat distribution is already

Completely nonholonomic: another tower, not necessarily of subbundles in $M$, just modules of vector fields tangent to $M$,
the Lie flag $D_1 = D$, $D_{k+1} = [D, D_k]$, $k = 1, 2, 3, \ldots$, growing in linear dimensions (pointwise), possibly slowly and depending on a point, but until the maximal value $\dim M$.

Observation/exercise. $D$ Goursat $\Rightarrow$ $D$ completely nonholonomic, some Lie algebra needed, not much, Jacobi identity...
Just one example, \( M = J^3(1, 1) \), the jets of order 3 of functions 
\[
( t, x(t), x'(t), x''(t), x'''(t) ) = ( t, x, x_1, x_2, x_3 ),
\]
\[
x' = \frac{dx}{dt} \iff dx - x'dt = 0
\]
\[
x'' = \frac{dx'}{dt} \iff dx' - x''dt = 0
\]
\[
x''' = \frac{dx''}{dt} \iff dx'' - x'''dt = 0
\]
\[
\left\{ \text{Pfaffian equations' description of the Cartan distribution} \right\}
\]
\[
\mathcal{C} \subset TJ^3(1, 1), \forall k \mathcal{C} = 2.
\]

**Vector fields' description**

\[
\mathcal{C} = \text{span} \left( \begin{pmatrix} x' \\ x'' \\ x''' \\ 0 \\ 0 \end{pmatrix} \right) \iff \text{span} \left( \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} \right) = \text{span} \left( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)
\]

The derived flag of \( \mathcal{C} \):

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
x_1 & 0 & 0 & 0 & 0 \\
x_2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

\[
[\mathcal{C}, \mathcal{C}] \subset [\mathcal{C}, \mathcal{C}] = T^2J^3(1, 1)
\]

General feeling before 1978: \( D = \text{Goursat} \Rightarrow \mathcal{D} \equiv \mathcal{C} \subset T^2J^3(1, 1), \forall k \mathcal{C} = 2 \)

Not true – an earthquake in Paris 1978 / exclamation of prof. P. Lieberman:

\[
D = \text{span} \left( \begin{pmatrix} x_3 \\ x_3 x_1 \\ x_3 x_2 \\ 0 \\ 0 \end{pmatrix} \right), \text{ Gian-Kunprka-Ruiz, CRAS, 1978:} \\
D \text{ is Goursat, } \neq \mathcal{C} \subset T^2J^3(1, 1).
\]

Their original proof long & cumbersome. A much posterior proof in 1 line: the sequence of linear dim's of the Lie flag of \( D \) at \( 0 \in \mathbb{R}^5 \) is \([2, 3, 4, 4, 5]\) \( \Rightarrow \) the small growth vector of \( D \) at \( 0 \in \mathbb{R}^5 \) while the \( \text{sgrv}(\mathcal{C}) = [2, 3, 4, 5] \) at every point (the Cartan distr. on jets is homogeneous). This is the tip of an iceberg....
The sqrv's of Goursat distributions whenever

\[
\begin{align*}
\text{in dimension 5:} & \quad [2, 3, 4, 5] \\
\text{in dimension 6:} & \quad [2, 3, 4, 5, 6] \\
[2, 3, 4, 4, 5] & \quad \# = 2 = F_{2,3-3} \\
[2, 3, 4, 5, 5, 6] & \quad \# = 5 = F_{2,4-3}
\end{align*}
\]

All sqrv's of Goursat have been computed. In the underlying
dimension \( r+2 \) (flag's length \( r \)) their \( \# = F_{2r-3} \) (Fibonacci).

Over the past 20 years parallel theories are being developed,
with the jumps in dimensions in the derived flag always by 2
(special 2-flags), or always by 3 (special 3-flags), etc.

Quite a lot of sqrv's of special 2- (3- \( \cdots \) ) flags is by now known,
but not all!! Getting hold of all of them is central in

the singularity theory of special multi-flags.

Differential geometry encounters here combinatorics, algebraic
geometry \( \cdots \).

In the year 2004 the computation of certain two sqrv's of
special 2-flags in length 3: \([3, 5, 6, 7, 7, 8, 9]\) and \([3, 5, 6, 7, 8, 8, 9]\)
allowed to discern two local geometric behaviours of 2-flags
and to pinpoint a serious mistake in a scientific paper.